Application of quantum field theory in particle physics, gravity, condensed matter physics and cosmology

1.) Gravitation and cosmology

I have always been interested in gravitation. One of my earliest papers[1] showed how to calculate the induced fermion number in gravitational backgrounds. In the 90's, and early 2000's, with my PhD student, Ariel Edery, MSc students André Méthot and Jihène Bouchami and my postdoc Luca Fabbri, and including several undergraduate senior thesis projects and summer interns, I investigated the possibility that conformal gravity was an alternative theory to Einstein gravity[2]. More recently, in 2013, with my MSc student Jonathan Belletête, we discovered that within an expanding universe, it was mathematically possible to have configurations of energy and momentum that were everywhere nonsingular and do not violate the dominant energy condition[3]. These configurations were smooth deformations of the negative mass Schwarzschild-de Sitter exact solutions of Einsteins equations. Subsequently, with my MSc student Saoussen Mbarek, we demonstrated that the smooth, negative mass bubbles could be obtained with very physical matter, the energy-momentum of a perfect fluid[4]. On a different tack, thinking about very high precision experiments with gravity, I thought of an experimental situation where the interaction of gravity with a very sensitive detector could be used to determine the speed of gravity[5] with a rather modest experimental set up. I want to continue my investigations concerning negative mass, indeed, these configurations should have import on the inflationary epoch in the early universe. I am also interested generally in the interaction of gravity with matter, specifically with a view to precision measurements and the interaction of gravity with quantum mechanical matter.

Negative mass solutions and the inflationary epoch "gravitational plasma":

My research into negative mass solutions came about with the simple question, "What is the meaning of the negative mass Schwarzschild[6] solution to Einstein's equations?" I still do not know. Our experience tends to show that all classical solutions seem to have some meaning in the full quantum theory. Soliton solutions describe particle states in the quantum theory that are inaccessible to perturbation theory. Euclidean solutions, instantons, are important in describing tunnelling phenomena. The Schwarzschild solution, the so-called black-hole, is a solution of the vacuum Einstein equations, imposing just spherical symmetry. It has one parameter, the mass, which can take any value, positive, or negative. The negative mass solutions have a naked singularity, there is no event horizon cloaking the singularity away from the view of asymptotic observers. However the singularity is at a fixed spatial position. In that sense, it is not as pernicious as the singularity of the positive mass Schwarzschild solution. Here the singularity is on a full space-line hypersurface which is simply unavoidable once an observer has crossed past the event horizon. The negative mass singularity is in principle avoidable. In fact, we deal with this kind of singularity in electrodynamics all the time. Point like charges do not disturb us, as we are content with the understanding that as we approach them, the singular point-like behaviour is almost certainly smoothed out due to a smooth, non-singular but concentrated charge density. Why can we not do the same for the negative mass Schwarzschild solution by introducing a non-singular but concentrated energy-momentum distribution? In fact, this is not possible due to the positive energy theorem[7]. This theorem states that if the energy-momentum satisfies the dominant energy condition[8], in a space-time that is asymptotically flat, then the mass perceived by an observer at infinity, *ie.* the ADM[9] mass, must be positive. The dominant energy condition is very desirable and physical, and cannot be abandoned. In simple words, it imposes that the energy and momentum of the matter that has been introduced to remove the singularity, moves forward in time and slower than or at most equal to the speed of light. The ADM mass of a locally smoothed out negative mass Schwarzschild space-time is necessarily negative, as no changes are to be made to the solution at infinity, hence no energy-momentum can smooth out the singularity and satisfy the dominant energy condition. With J. Belletête, we realized that a space-time that is not asymptotically flat, such as de Sitter[10] space-time does not admit a positive energy theorem.

There exist exact solutions called Schwarzschild-de Sitter space-times, which correspond to a massive particle sitting inside a de Sitter space-time, and satisfy the Einstein equation in the presence of a cosmological constant. These solutions have two parameters, the mass and the cosmological constant, each can take any value, positive or negative. For positive cosmological constant, there is no positive energy theorem, and the negative mass solutions can be smoothed out to be non-singular with energy-momentum that satisfies the dominant energy condition. We have shown the existence of configurations that correspond to non-singular bubbles that look like negative mass in de Sitter space-time. We have not yet established the existence of solutions of the equations of motion of a dynamical system of some form of matter interacting with gravity that correspond to static, non-singular, negative mass bubbles. this is the first project that we wish to address. The **methodology** that we will use is to consider a thin wall configuration that interpolates from a finite ball of negative mass energy-momentum inside, interpolating to empty de Sitter space-time outside. At the wall, we will impose the Israel junction conditions[12], which guarantee conservation of energy and momentum across the wall. This method is well established and it is clear that the matter content will have to be of a special type. The negative mass bubbles found in [4] have negative pressure that interpolates from zero at the centre to a negative constant outside, while the density increases from zero at the centre to a finite positive constant outside the bubble. Thus the solutions interpolate from essentially flat space to de Sitter space. As the work [4] shows, the negative mass bubble should be thought of simply as a deformation of the background vacuum energy density, that corresponds to a local region of under density. Such a configuration may energetically wish to expand and dilute to infinity or collapse to a singular configuration. However, such instabilities are common in the study of solitons, and it is well understood how to add terms to the energy, which depend on gradients, so that we can stabilize the solitons. Thus, it should not be surprising that the existence stable, static solutions with negative mass will be easily established. The next aspect of the project that I would think about is the consequences of negative mass to the inflationary epoch. Indeed, the energy is still conserved, so one cannot imagine the creation of arbitrary amounts of negative mass, lowering the total energy, and corresponding to a basic instability. The negative mass must be created at the same time as positive mass. Furthermore, there exist theorems that classical, normal time evolution from a initial configuration of positive mass, can never yield any region of negative mass[8]. However these theorems do not forbid the possibility of tunnelling. Indeed, pair creation of black holes has already been considered in [13] in the inflationary universe. The important point is that for pair creation to work, the objects created must not fall back together. In the case of black holes, their mutual attraction must be compensated for by the rate of expansion of the universe, rendering the process almost negligible, except for Plank scale inflation. Our methodology will be to adapt the existing calculations to the case of the production of two negative mass particles, or a negative-positive mass pair. Two negative mass particles feel a mutual repulsion, and for a positive-negative mass pair the positive mass attracts the negative mass particles, while the negative mass repels the positive mass particle [3,4,14] yielding a situation in between repulsion and attraction, where the negative mass particle chases after the positive mass one. Since the evolution of the two particles after creation will not tend to their annihilation, we expect that the rate of production will be correspondingly enhanced. These are the short term goals of this project.

The **long term vision** of this project, corresponds to the application of the negative mass solutions to the cosmological evolution of the universe. We can imagine that during the inflationary epoch of the evolution of the universe, there is actually a plasma of positive and negative mass particles. This strange gravitational plasma has never been investigated. It should have a significant effect on the propagation of gravitational waves. The analogy between electromagnetic waves and a plasma of charged particles does not hold, since there, the superimposed electric fields of the wave have opposite effects on the charged particles of the plasma, the positive charges are pushed one way, the negative charges are pulled the other way. For a gravitational wave, because of the equivalence principle, all particles are pushed the same way, be they of positive or negative mass[3,4,14]. Thus it becomes unclear whether the effect of the wave on the plasma can be phenomenologically summarized by an induced polarization. In the electromagnetic case, the induced polarization tends to reduce the applied electric field[15], with the gravitational plasma, the corresponding susceptibility might in fact be negative. Instead of using standard plasma physics techniques, my methodology for this problem would be to analyze instead the partition function of the plasma. This would correspond to a 3-dimensional Coulomb gas superimposed on de Sitter space-time. There is a well understood description of the Coulomb gas partition function in terms of a sine-Gordon model[16] in flat space. Mapping the problem to the sine-Gordon theory makes many things obvious, for example the existence of a Debye length and the corresponding screening of waves below the plasma frequency. The generalization to the de Sitter context should not be particularly difficult, in a first approximation one would simply get the sine-Gordon model in the inflationary universe. A renormalization group analysis[17] of the Coulomb gas can be used to find the behaviour of correlation functions and possible phase transitions (although these seem likely only in the 2-dimensional context). What is the most important question is to find a directly observable signal that would be a necessary consequence of this early inflationary phase. Such an observable would be similar to the possible observation of B-modes and the BICEP experiment[18] (now debunked[19]) . In that respect, the inflationary epoch should be unstable to creation of negative-positive mass particle pairs. However, these being created because of the existence of the background vacuum energy density, should have a back reaction on that energy density, tending to decrease it. This is the expected behaviour of a stable system and goes by the name of Lenz's law[15] (in the electrodynamical context). Thus the back reaction of the pair creation could in fact give rise to a mechanism for the graceful exit[20] from the inflationary epoch.

The early universe almost certainly passed through the phase of inflation. Therefore it is strongly suggested that there were density fluctuations in this epoch that resulted in localized distributions of negative mass. The presently observed voids[21], the largest structures in the universe, could well be explained to have been generated by regions of negative mass. These negative mass objects must be in equilibrium with an equal amount of positive mass objects, and the system would behave in some sense like a neutral plasma. The actual phase of the plasma would depend on conditions such as the temperature and the density. The relics of this strange gravitational plasma on the present universe, especially on the microwave background must be uncovered. The actual observational proof of the existence of an inflationary epoch in the early universe is of staggering significance. It would give a confirmation of ideas of the models grand unification[22] of particle physics.

Since my MSc students have moved on, I have been working on this project essentially by myself. Ideally I would like to hire a postdoc to collaborate with. If that is not possible with the funding that I receive, I would like to engage a PhD student to work on the project. I am presently interviewing Christian Quirouette of the University of Ottawa, among others, concerning working on this project.

Gravitational interaction with matter.

My first investigation in this line corresponded to a method to measure the speed of gravity[23]. My interest was stirred in precision tests of gravitational phenomena after I attended the Moriond meeting on gravitation in spring 2011[24]. Many researchers spoke of their experiments, which involved moving sources and no one seemed to worry about the finite speed of propagation of the gravitational force. I thought of an experimental set up where a signal would occur due to the finite speed of propagation of gravity. Almost certainly this speed is the same as the speed of light. However not a single experimental measurement has yet to measure any dynamical aspect of Einstein gravity. Indeed, a huge and expensive experimental activity has been set up to measure gravitational waves with the LIGO project[25]. However just to confirm that gravity is a dynamical force that does not instantaneously in the Newtonian action at a distance way, would be extremely important. I thought of the following setup, a detector is placed along the line joining two masses at unequal distances from the detector. As the masses each attract the detector, there is a point where the net force on the detector is zero, take for concreteness a mass M at distance d and quarter of the mass $M/4$ at half the distance $d/2$. Clearly Newton's law makes the force on the detector cancel. If the masses are moved, simultaneously and quasi-statically, outwards and inwards by a tiny fraction of the initial distance, proportionally as in ϵd for the mass at distance d but $\epsilon d/2$ for the other, then again, at each instant, the force cancels. However, if the masses are oscillated with a frequency ω, then the force at the detector does not cancel since the change of the force from the closer mass arrives at the detector faster than the cancelling change in the force from the further mass. The resultant force at the detector is periodic $\sim \sin\omega t$, but with amplitude proportional to the Newtonian force of just one of the masses multiplied by frequency ω, the fractional amplitude of the oscillation ϵ and the relative time delay Δt. With $\omega 1 \sim 10^3$ Hz, $\epsilon \sim .1$ and $\Delta t \sim 10^{-8}$ reduces the Newtonian force by a factor of 10^{-6}. The gravitational force of a macroscopic object on the detector can be of the order of 10^{-11}Newtons, which would be the force of a 1Kg mass at 1 metre distance. Such a force was measured over 200 years ago by Cavendish. Surely experimental precision has improved in the intervening time that a million fold decrease in the force is measurable. Indeed, Kasevich[26] using atom interferometry can measure $\Delta g/g \sim 10^{-15}$, where g is the earth gravitational acceleration, which is just shy of the precision required. I have taken Alexandre Landry as a PhD student to continue my research in this field. We are studying three different physical systems. First, the possibility of inducing transitions in milli-Kelvin neutrons using oscillating macroscopic masses. Milli-Kelvin neutrons correspond to a physical system that can be easily isolated from all physical interactions except gravitational. The experiment q-Bounce[27], observes that the neutrons occupy the Schrodinger levels corresponding to a particle in a linear potential mgh corresponding to the earth's gravitational potential. The wave functions are Airy functions and the energy levels are separated by pico-eV. In the experiment q-Bounce, transitions between the quantum levels were induced through the coupling to the oscillations induced in the floor of the vessel, and indeed transitions were observed. However, we propose to induce the transitions through stimulating the neutrons with gravitational interactions. simply oscillating a mass in the presence of the neutrons at the resonant frequency (which is a few hundred Hz) would provoke transitions. We are studying the experimental feasibility of the proposed experiment. Second, we are looking at the effect of oscillating masses on quantum mechanically entangled wave functions. The Kasevich[26] atom interferometer experiment corresponds to a single Rubidium atom that is put into an entangled state through Raman or Bragg scattering of a photon off the atom. There is a finite probability that the photon scatters off the Rb atom, if it does, the atom rises up relative to the the unscattered wave function. In the end, since the photon scattering or not is a quantum event, the atom enters into an entangled state of these two possibilities. The the phase of the wave function that is higher in the earth's gravitational potential changes differently than the one that is lower, and when brought back together, an interference is observed. Our idea is to induce differential changes in the phase of the two parts of the entangled wave function through time dependent oscillating masses. Indeed, the same principle can be used as in the experimental setup proposed to measure the speed of gravity above, but at the level of the gravitational potential rather than the force, to suggest another method to measure the speed of gravity. The one subtlety is that the gravitational potential of physical masses is always negative. Thus we cannot imagine that the potential can ever cancel at a given point unless we can have negative masses. This may seem impossible, however, it is in fact easy to construct negative masses in ordinary systems. Imagine hollowing out a non-centred spherical cavity inside a larger sphere of uniform density. Obviously the gravitational potential of this body will be the sum of the potential as if the larger sphere has no cavity plus the potential of the cavity but now filled with negative mass. Now spinning the sphere with the cavity about its centre, produces a time dependent potential as if only the negative mass body were moving. If the same is done on the other side of the atom interferometer with an identical sphere, however now with the spherical cavity filled with a material with twice the density of the larger sphere, we obtain a situation where the time dependent potential cancels at one point, but not elsewhere. One could repeat the experiment described above to measure the speed of gravity by arranging for the gravitational potential to cancel at the location one of the wave functions, but not the other. There are presumably many other experimental possibilities that would lead to interesting measurements of the interaction of quantum mechanical wave functions with gravitational fields. We are presently thinking about these ideas.

2.) CP-Violation and solitons
CP-violation in particle physics is a very small effect, obtained by the introduction of the CKM matrix with complex phases, in the mass matrix of the quarks[28]. However the amount of CP violation that is required to account for the baryon asymmetry[29] of the universe is much greater than what the CKM matrix provides. A new source of CP violation is necessary. In this project we try to find non-perturbative sources of CP violation, within the standard model. Our idea is to look for contributions to CP violation that comes from solitons and instantons, which are normally non-perturbative in the coupling constants. So far, this is a small project that I have pursued in collaboration with two visiting students, R. Srivastava and N. Chandra, who came from India for short visits, with the Canadian Commonwealth Bursary Program, a program that was sadly cancelled about two years ago. We have studied an effective model of CP-violation that could be used to describe the decay of one heavy scalar mesons to 4 lighter scalar mesons. This corresponds to an effective 5 scalar field interaction that is CP violating. The simplest interaction that we can write is of the form

$$\sim \lambda\epsilon^{abcde}\epsilon^{\mu\nu\lambda\rho}\phi_a\partial_\mu\phi_b\partial_\nu\phi_c\partial_\lambda\phi_d\partial_\rho\phi_e$$

This effective interaction is CP violating if the fields ϕ_a are scalar fields (not pseudo-scalar). Adding appropriate kinetic terms and the higher derivative terms for the scalar fields yields stable, non-trivial, Q-ball[30] type solitons solutions for the fields. If the fields are taken to be pseudo-scalar, then this term is not CP violating, however it is the first term in the expansion of the Wess-Zumino term for the Skyrme model[31] to lowest non-trivial order in the meson fields. It is actually quite interesting that the Skyrme model admits solitons that are not Skyrmions. The solitons that we find are only numerical solutions. We are working out the non-perturbative contributions to CP violation and particle physics in general that such solitons represent. In a 1+1 dimensional analogous model, we are able to find exact analytical solutions corresponding to Q-ball type solitons and also the corresponding Euclidean solutions called instantons. The Lagrangian density that we consider contains three fields

$$\mathcal{L} = (1/2)\partial_\mu\phi_i\partial^\mu\phi_i - g(\phi_i\phi_i - a^2)^2 - \lambda\epsilon^{ijk}\epsilon^{\mu\nu}\phi_i\partial\mu\phi_j\partial_\nu\phi_k.$$

It is quite amazing that this model yields equations of motion that are analytically integrable. The model could have some application in one dimensional condensed matter systems. I plan to continue studying this system with Jorge Gamboa and Fernando Mendez of the University of Santiago, Chile, where I plan to visit on sabbatical leave in two years. I believe that two publications will be forthcoming, we have almost finished the work leading to them.

3.) Induced, false vacuum decay
We have studied induced false vacuum decay due to topological solitons. False vacua arise in many physical and theoretical concepts. A super heated or supercooled liquid is in a false vacuum state, the universe during the inflationary epoch is theorized to be trapped in a false vacuum, and the only cosmological solution[32] in string theory corresponds to the universe being trapped in a false vacuum. A false vacuum is intrinsically unstable to decay via quantum tunnelling and topological solitons are particularly suited to provoking the decay. This is because these solitons necessarily contain a zero of the scalar field in their interiors, a point where the symmetry is unbroken and hence the true vacuum. Thus a spontaneously broken, false vacuum would naturally be unstable if there existed a non-trivial vacuum structure in which these topological solitons can arise. We showed that when the solitons are thin walled, they can be classically stable, but quantum mechanically meta-stable due to quantum tunnelling. We have considered topological solitons corresponding to monopoles, vortices/cosmic strings and domain walls[33] giving rise to 5 papers. Future work will involve taking into account gravitational corrections. We also wish to generalize our results to include Skyrmion like solitons,

which are based on different topological considerations. It is of tremendous importance to understand the stability of the false vacuum. The only cosmological solution of string theory corresponds to the possibility that the universe is in a false vacuum[32] , that will not decay in the foreseeable future. The viability of string theory then is brought into question if the decay rate is found to be appreciable.

The next phase of this research project is to take into account gravitational corrections. Indeed, in a cosmological context, gravitational interactions cannot be neglected. The original papers of Coleman, Callan and de Luccia[34] considered false vacuum decay due to instantons, but then addressed the question of gravitational corrections. The **methodology** that we will use for this study is clear. Since the solitons that were considered were always thin walled, we will continue that approximation. Then in the gravitational context, the conservation of energy-momentum across the wall gives rise to the Israel junction conditions[12]. We will apply these conditions to the topological solitons that we have already considered in the absence of gravitation. The expected scenario will be that the space-time is in the false vacuum outside the solitons, and hence in a de Sitter space-time. Inside the soliton, because it is close to the true vacuum, the space-time will be close to Minkowski. The wall separating the two space-times will have a surface tension that will exactly balance the pressure difference between the two space-times. But more generally, simply the vacuum energy difference between the inside and the outside of the soliton should be positive, giving rise to all different possibilities. Additionally, for monopoles, the exterior space time has to be taken Reissner-Nordstrom[35] to take into account the mass and the magnetic charge of the bubble, while for the cosmic string, some kind of conical space-time with a defect angle should be appropriate. For weak gravity, we do not expect a great change in the tunnelling amplitudes. However for strong gravity, there could be important corrections. I expect to supervise a MSc student for this project. At the moment, Éric Dupuis, who is R. MacKenzie's MSc student, is in fact going to be co-supervised. Collaboration with M. Haberichter, a postdoc at Kent, will also continue.

4.) Quantum spin systems

Large spin quantum systems can be analyzed using the spin coherent state path integral which we have reviewed in our Physics Reports[36]. A quantum spin is described dynamically by a potential, which originates in the local surroundings of the spin in a crystal lattice, or external forces such as applied magnetic fields, and a kinetic term which is called the Wess-Zumino term for spin systems, and neatly takes into account the torque on a spin which is orthogonal to the direction of its motion and the applied force. We have computed tunnelling with this path integral in several systems resulting in the publications[37], finding the corresponding instantons. We have shown that high orders in perturbation theory can be efficiently computed using the path integral. The most important result that we have found corresponds to identifying the ground state of an N site, quantum spin chain, with nearest neighbour exchange coupling and an on-site easy-axis anisotropy. Here we found that for even or odd number of sites, the ground state is radically different. The odd spin chain with an anti-ferromagnetic coupling must have a defect. In the limit that we studied, the defect is localized between two spins, however, we expect that the defect becomes a more spread out soliton as we vary the coupling constants. To study this process further, with my student Christian Boudreault we are studying the related Blume-Capel-Haldane-Ising (BCHI) model $H = \sum_{i=1}^{N}\left(a(S_i^z)^2 + bS_i^z S_{i+1}^z\right)$. This is the model generalized to arbitrary s of Blume and Capel (which was for spin 1) that figures as the anisotropy in the Haldane model[38]. In his paper, Haldane made the famous conjecture that the spin chain for integer spins is gapped, while for half odd integer spin it is gapless. The Haldane model should be studied over the entire domain of its parameter space.

In our Hamiltonian, since only S_z appears, our Hamiltonian is trivially diagonal. However, it does not automatically tell us which is the ground state. We find the interesting result that the solitons have a finite size, that depends only on the ratio of the two coupling constants a and$b$ and not on the number of sites. We find our results in the case of the odd spin chain, however, the size and properties of the soliton depend only on the coupling constants.

Therefore, the solitons would be identical in the model with an even number of spins or with open boundary conditions. In the limit $a \approx b$, the solitons tend to be massless. We are in the process of completing two papers on this model. Study of the BCHI Hamiltonian for arbitrary values of the coupling constants is very important, as it is the term that defines the anisotropy that chooses the ground state of the Haldane model. The Haldane conjecture, that the integer spin chain is gapped while the half odd integer chain is not, is an open problem for more than 30 years. A full understanding of the Haldane model its phase structure and its spectrum is of great interest. Our **methodology** in this research is to use rigorous mathematical analysis, to analyze the spectrum and the ground state of the Hamiltonian. This approach will be augmented by perturbative calculations and non-perturbative tunnelling amplitudes obtained from the path integral and instantons and the spin-coherent states path integral. Our previous work has used all of these methods quite successfully. We plan to continue with the study of the Affleck model of dyons of fractional charge on the Haldane spin chain, with a Peierls instability[39]. We plan to find the solitons of this model and make the connection with the BCHI solitons that we have found. The **long term vision** of this project would be to address the question of the Haldane conjecture and to map out the entire phase diagram of the Haldane model over its whole parameter space.

5.) Anyons and the virial expansion:
We have developed a method to numerically simulate (hard sphere) anyons on the lattice, and we have been able to compute the expectation values of various operators, such as the Wilson loop, the 't Hooft loop and the energy. Our method allows for a simple generalization to computing the canonical partition function for N anyons, and hence the virial coefficients. It is an open problem if the virial expansion is useful for a gas of anyons. Only the first few coefficients have been accessed theoretically, and it is not known if the equation of state admits a useful expansion in the virial coefficients. Anyons could be used for simulating quantum computation which would have significant impact. This project requires me to hire a PhD student with experience in computing. The project will also be done in collaboration with my ex-PhD student, Faiza Nebia-Rahal, who is now a professor in the CEGEP system.

We have been studying a Monte Carlo numerical simulation of an effective description of the symmetry broken sector of the 2+1 dimensional abelian Higgs model. We performed our simulation on very special lattice, in the limit of strong coupling. In this limit, the perturbative massive particles are decoupled and the only particles that are left are the topological solitons known as Nielsen-Olesen vortices[40]. The vortices are also essentially non-interacting however, they do however have a zero range repulsion, ie., they have a delta function repulsion between them. The numerical simulation, to compute the vacuum to vacuum amplitude, then corresponds to the summing over all possible intermediate configurations which are vortex anti-vortex pairs in Euclidean space, describing closed vortex loops. The loops have a Euclidean action that is in first approximation equal to the vortex mass times the loop length. The publication of our work[41,42] corresponds to the first numerical simulation of anyons[43]. We highlight that our paper contains a very novel fashion to obtain a set of closed loops, our lattice corresponds to a tetrahedral filling of Euclidean space with non-regular tetrahedra and we put as variables the cube roots of unity on the vertices. Here the Chern-Simons term adds two complications to the simulation. The first is that the Euclidean action is no longer real, the Chern-Simons term remain imaginary upon analytic continuation to Euclidean space. This precludes the use of the exponential of the Euclidean action as a probability distribution in the Monte Carlo procedure, since it is no longer a real number between 0 and 1. However, it is permissible to use only the real part of the Euclidean action to generate a set of equilibrium configurations and which actually defines the measure on our space of configurations, and then simply integrate the Chern-Simons term as a uni-modular phase against the measure so defined. The second complication is that the computation of the Chern-Simons term which is just given by the total linking number of all the vortex loops, is very hard. There exists the Gauss linking number formula requires the calculation of two line integrals over the positions of the two linked curves, but this integral is not feasible and prone to numerical approximation errors. We actually compute the linking number using some ideas of knot theory[44], which was a tour de force, we refer you to the published article for the details[42]. Our paper examines the Wilson loop and the 'tHooft loop in the anyonic theory. We found the surprising result that both of these exhibit the perimeter law, even though there are no massless particles in the theory.

For the future, I propose to study the anyon gas in more detail, numerically. We can compute the N anyon canonical partition function. It should be no problem to compute this numerically for N from $1, 2, \cdots \sim 100$. The N particle canonical partition function is used to calculate the Nth order virial coefficients[45]. The virial coefficients define the equation of state through the relation: $PV/nK_bT = \ln \Xi = \sum_i (n/V)^i B_i$ where Ξ is the grand canonical partition function and $B_i,\ i \geq 2$ is the ith virial coefficient. For example $B_2 = (1/2 - Q_2/Q_1^2)$ where Q_i is the i particle canonical partition function. The higher terms are increasingly complicated functions of the N particle canonical partition function. The second virial coefficient of anyons has been computed using analytic approximations[46] and shows interesting non-analytic behaviour. Some approximate calculations of the third virial coefficient exist and a perturbative expansion in the statistics parameter for all the coefficients exists[47], but exact coefficients are not known. With our method for generating anyons, we can compute the N particle canonical partition function by doing a simulation in periodic Euclidean time, and then easily compute the virial coefficients for any value of the statistics parameter. Our more ambitious goal is to try to simulate non-abelian anyons. Such objects correspond to degenerate multiplets of anyons, however now the wave function of the degenerate multiplet changes by a matrix valued unitary representation of the braid group upon exchange. We imagine that decorating our spin variables with additional degrees of freedom could render our anyons non-abelian. Quantum simulation of non-abelian anyons could shed light on topological quantum computation[48]. Our **methodology** is clear, we will perform Monte-Carlo lattice simulation. The innovation and originality in our proposal is also clear, no one else has been able to do any simulations of anyons on a lattice.