The quadrupole formula: a 100 years later

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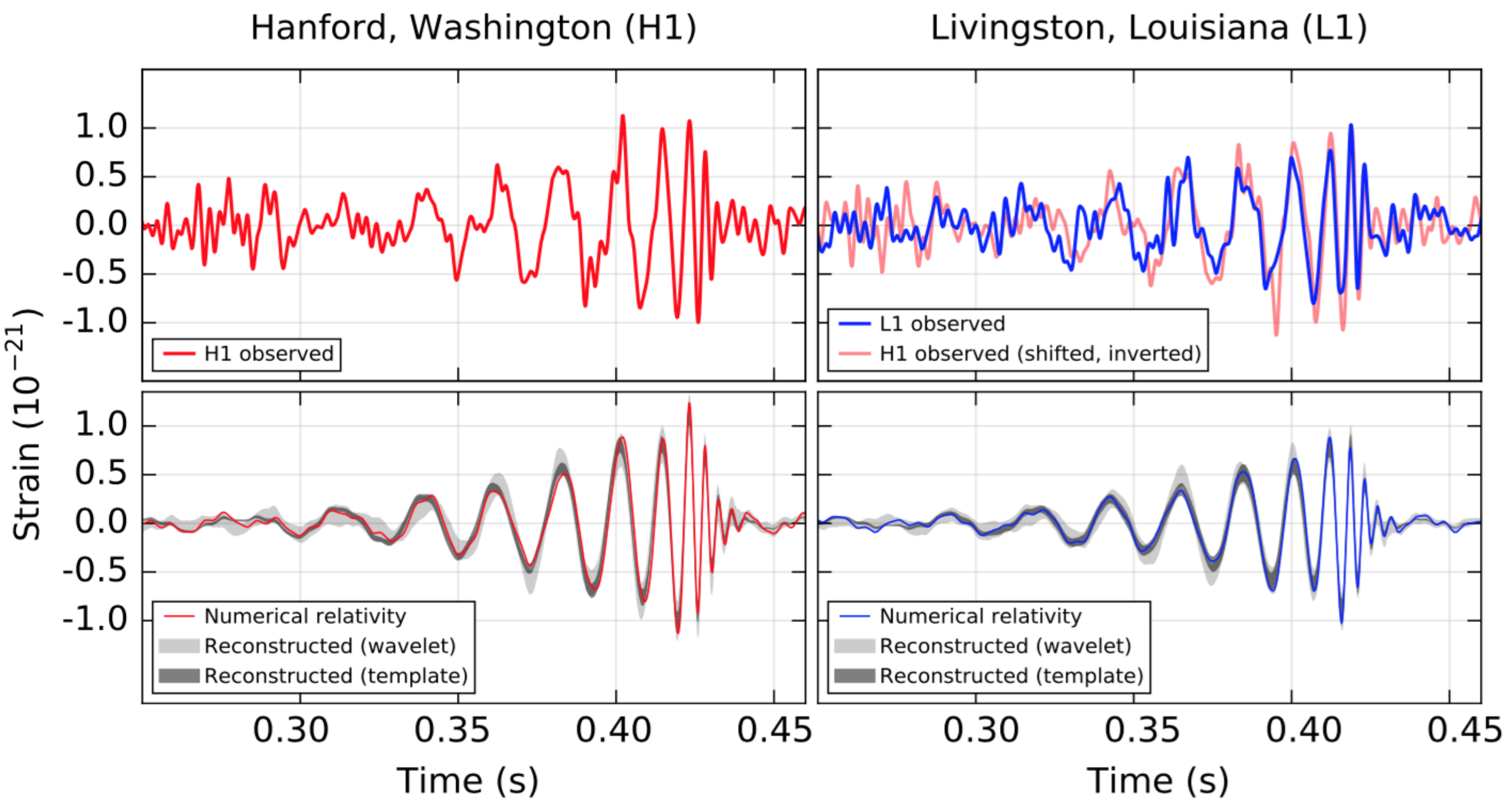
Gravitational radiation



[LIGO Scientific Collaboration and Virgo Collaboration, PRL 2016]

Some history

1916
Einstein predicts gravitational waves (GWs)

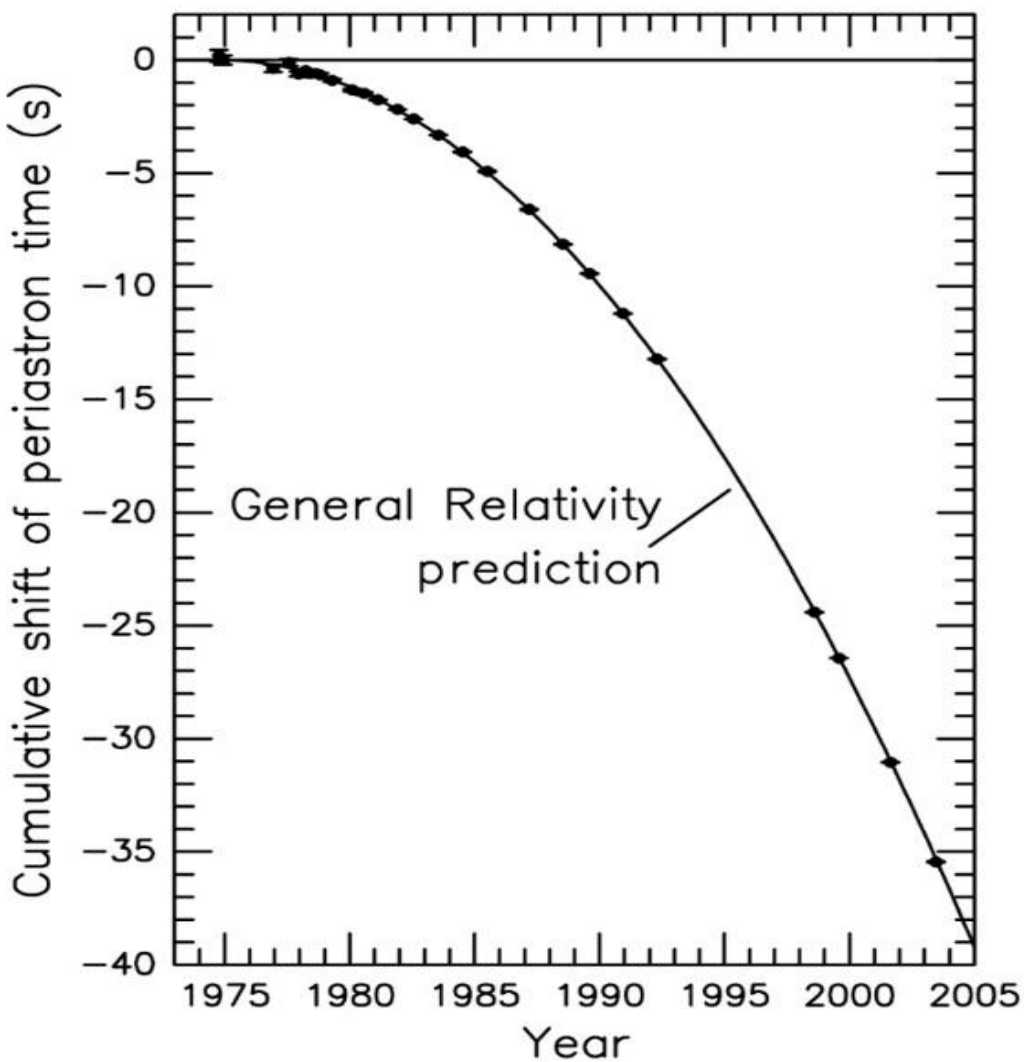
1960s
Bondi et al: "They are real!"

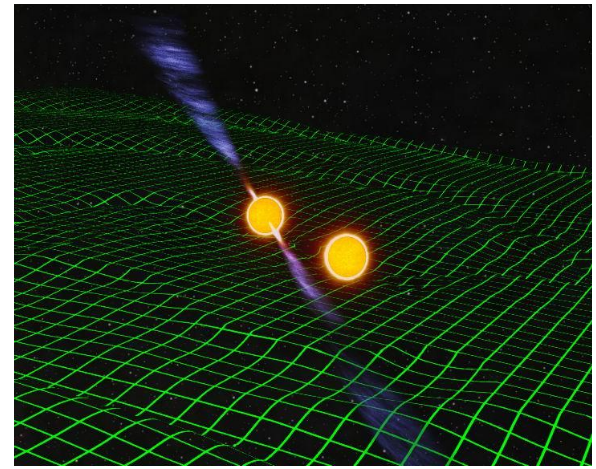
Hot debate whether GWs are real

1974
Observation by Hulse and Taylor confirms this

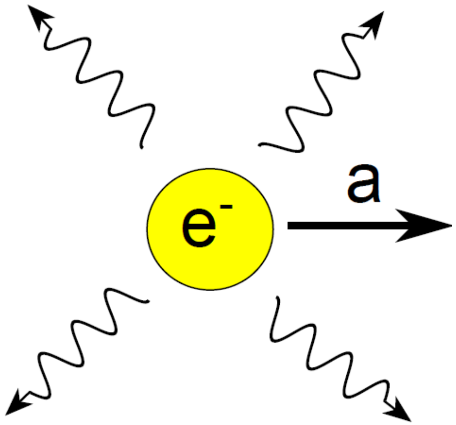
Sept 14, 2015
First *direct* detection by LIGO

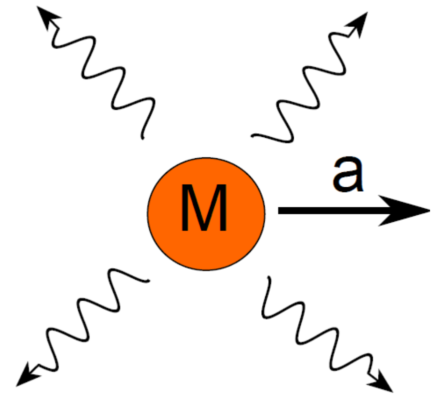
Hulse-Taylor pulsar





Quadrupole radiation





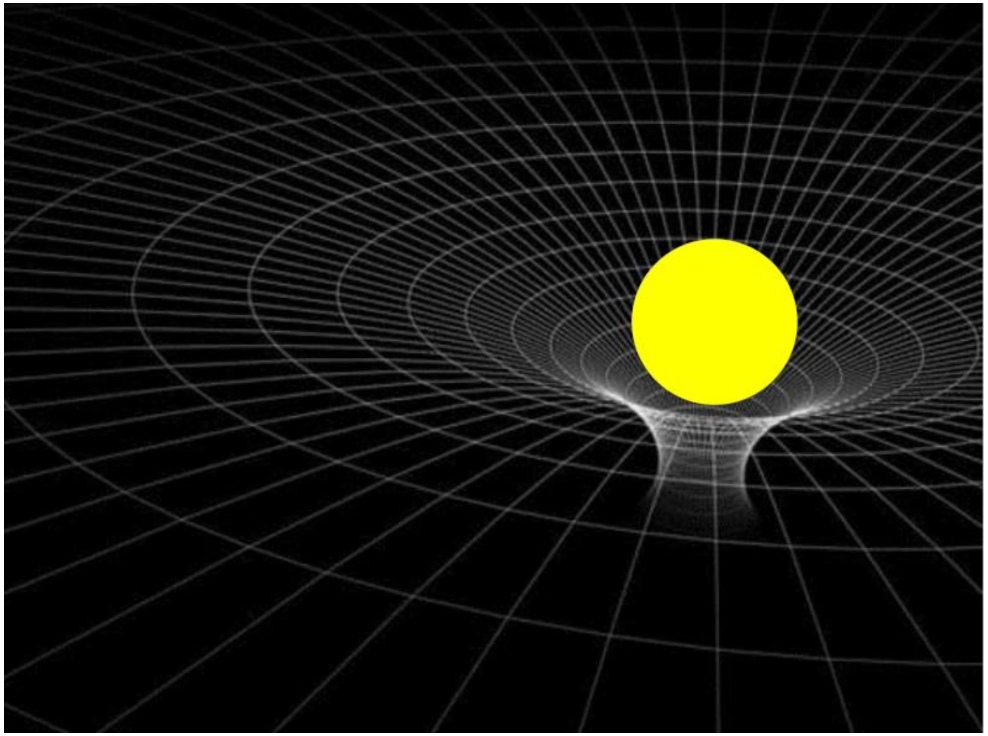
$$P = \frac{\mu_0}{6\,\pi\,c}\,\ddot{p}_i\,\ddot{p}^i$$

$$P = \frac{G}{5\,c^5}\,\dddot{Q}_{ij}\,\dddot{Q}^{ij}$$

Charge/mass conservation → no monopole radiation

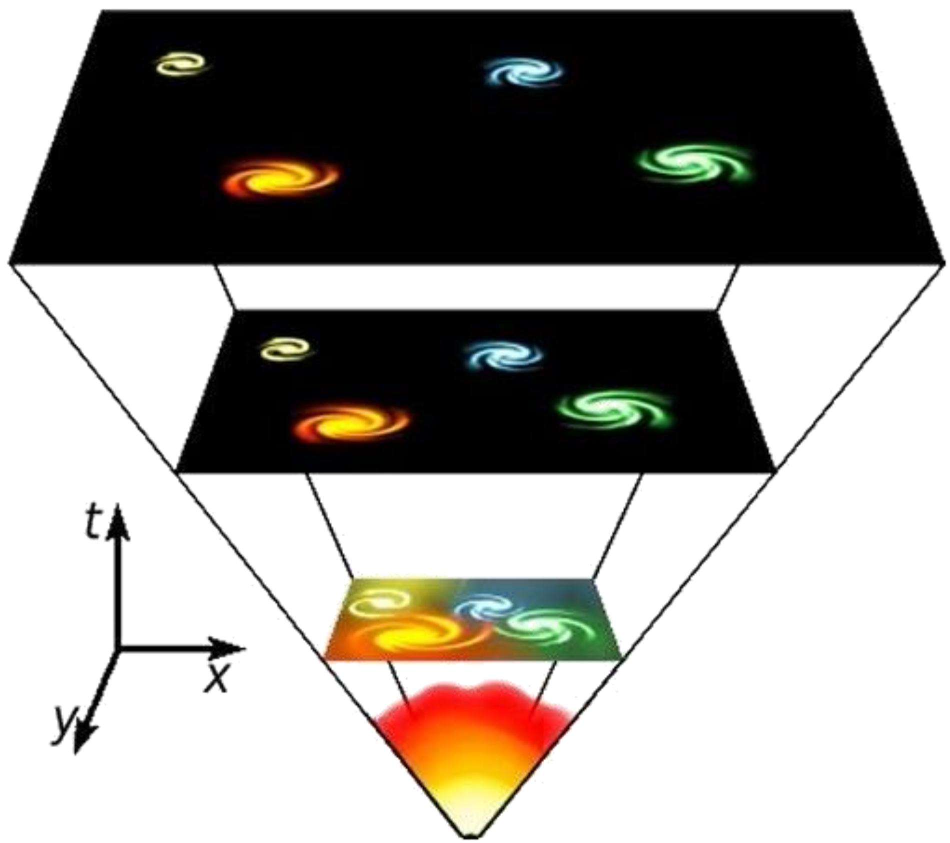
Momentum conservation → no dipole radiation

Critical assumption



Move far away from sources: ‘spacetime becomes flat’

Expanding spacetimes are not asymptotically flat!



Why assume asymptotic flatness?

Conference Warsaw 1963

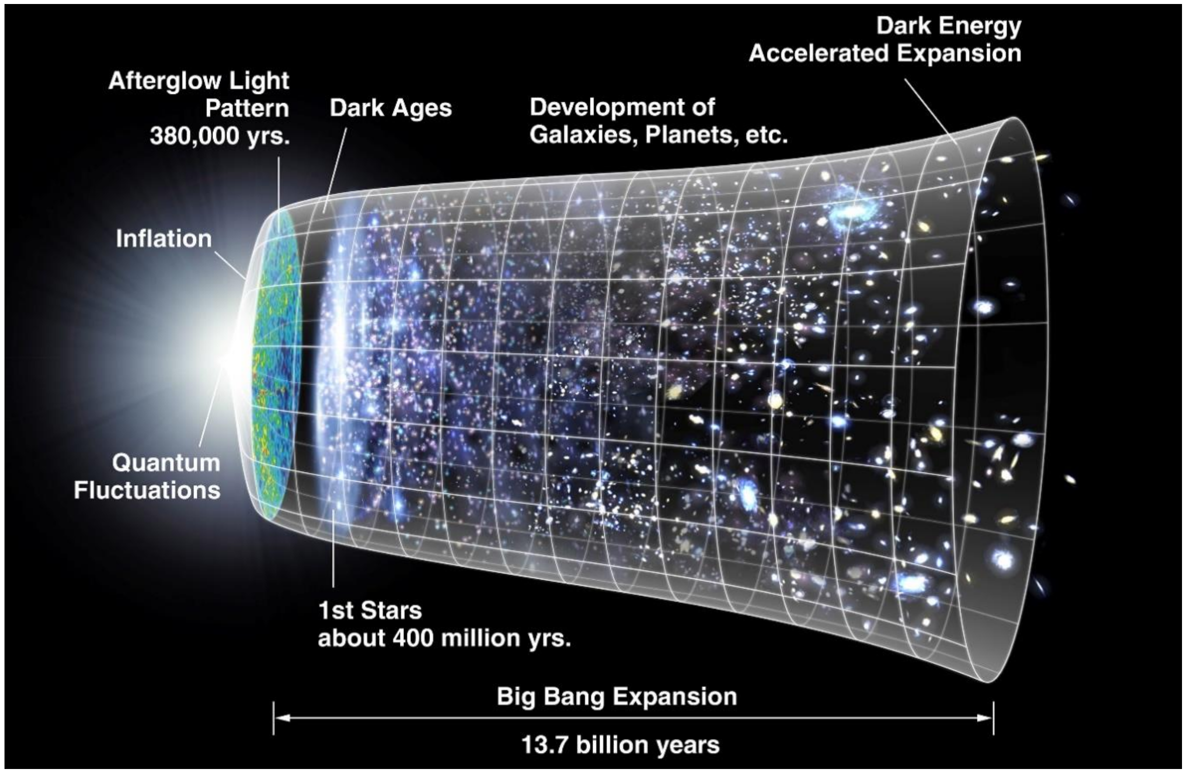
P. G. BERGMANN:

The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. BONDI:

I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

Modelling the expansion

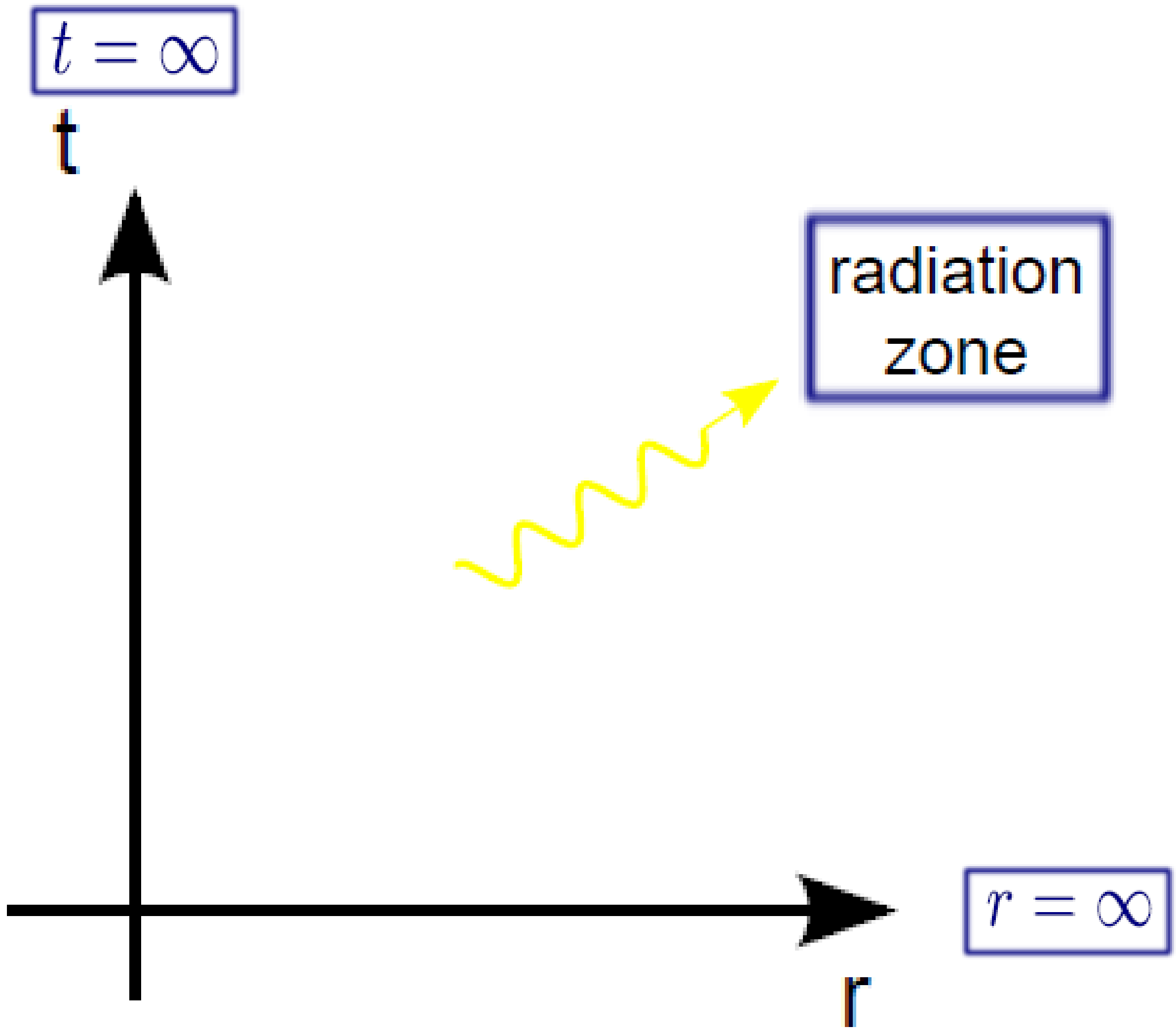


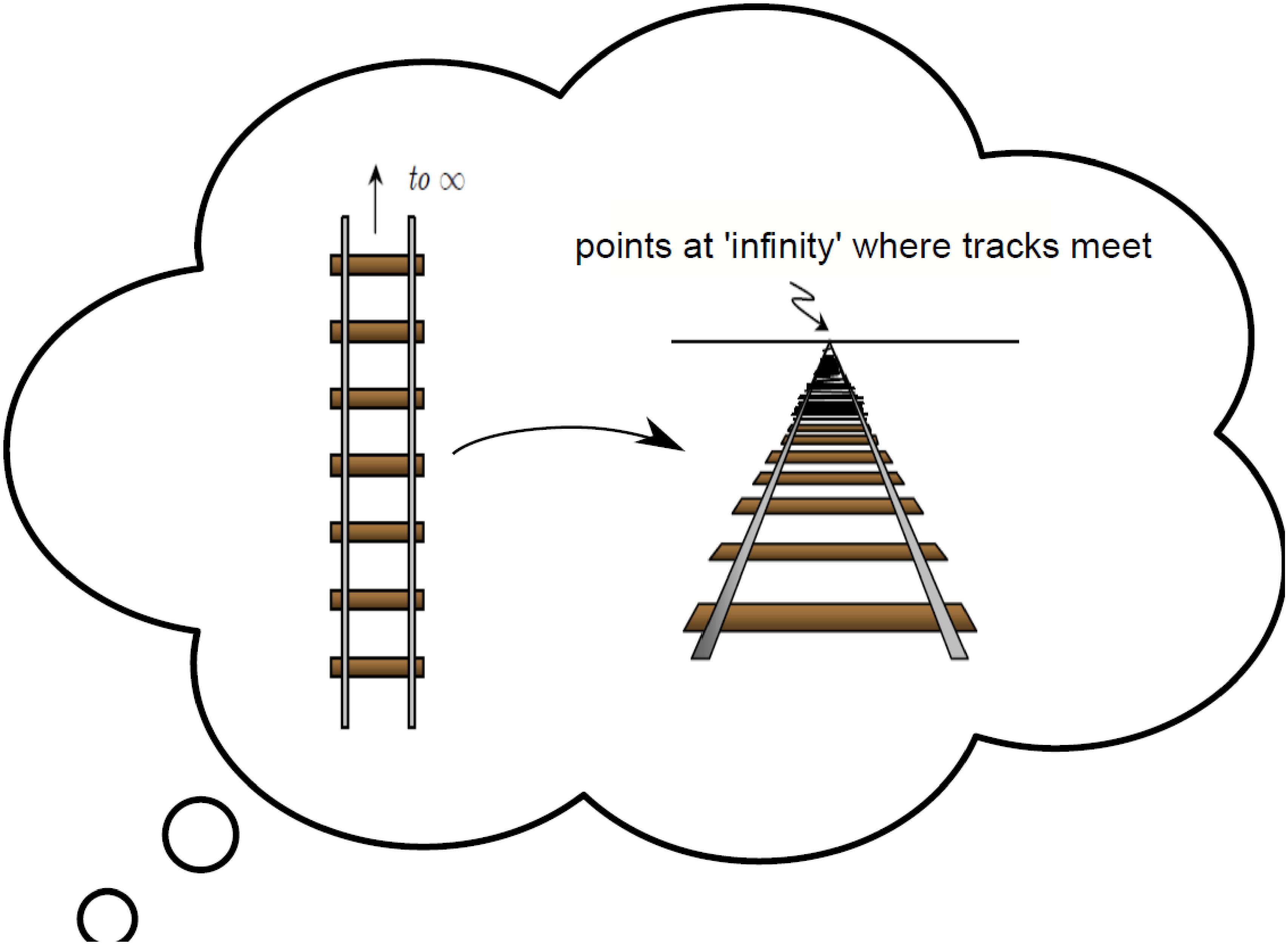
Describe the Universe by a de Sitter spacetime (= vacuum with a cosmological constant Λ)

But isn't Λ small?

Even though $\Lambda \sim 10^{-52}\ m^{-2}$, it can cast a long shadow!

Intermezzo: conformal diagrams

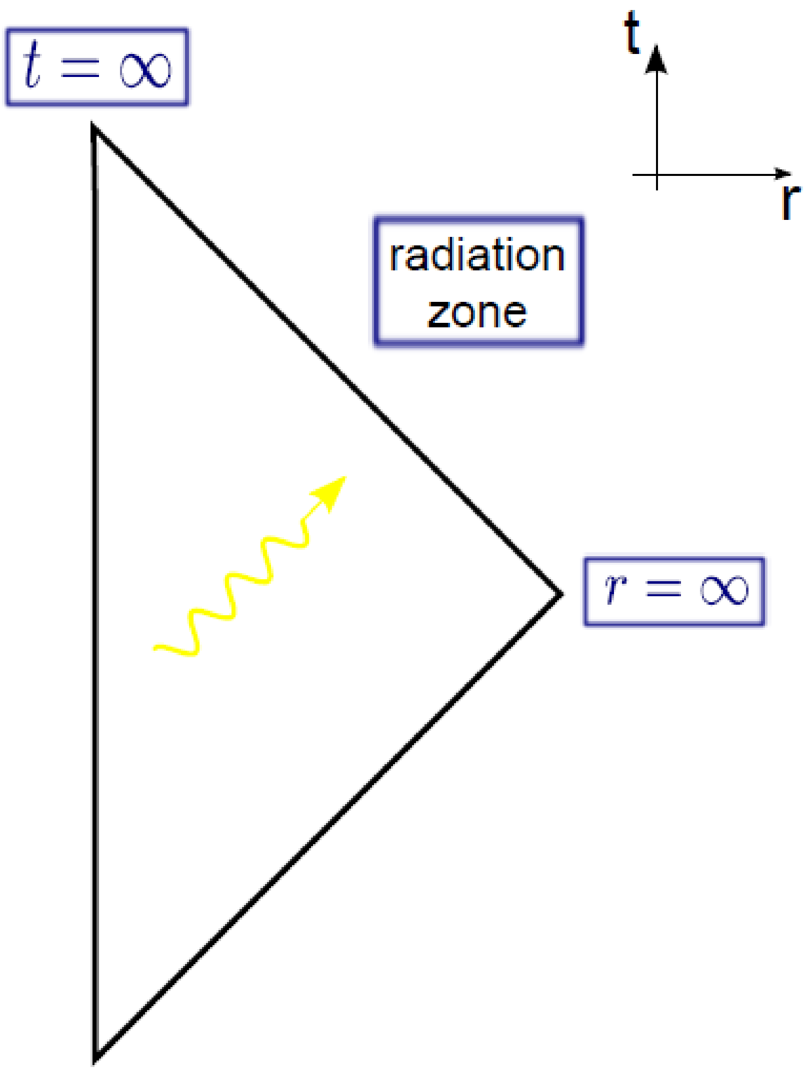




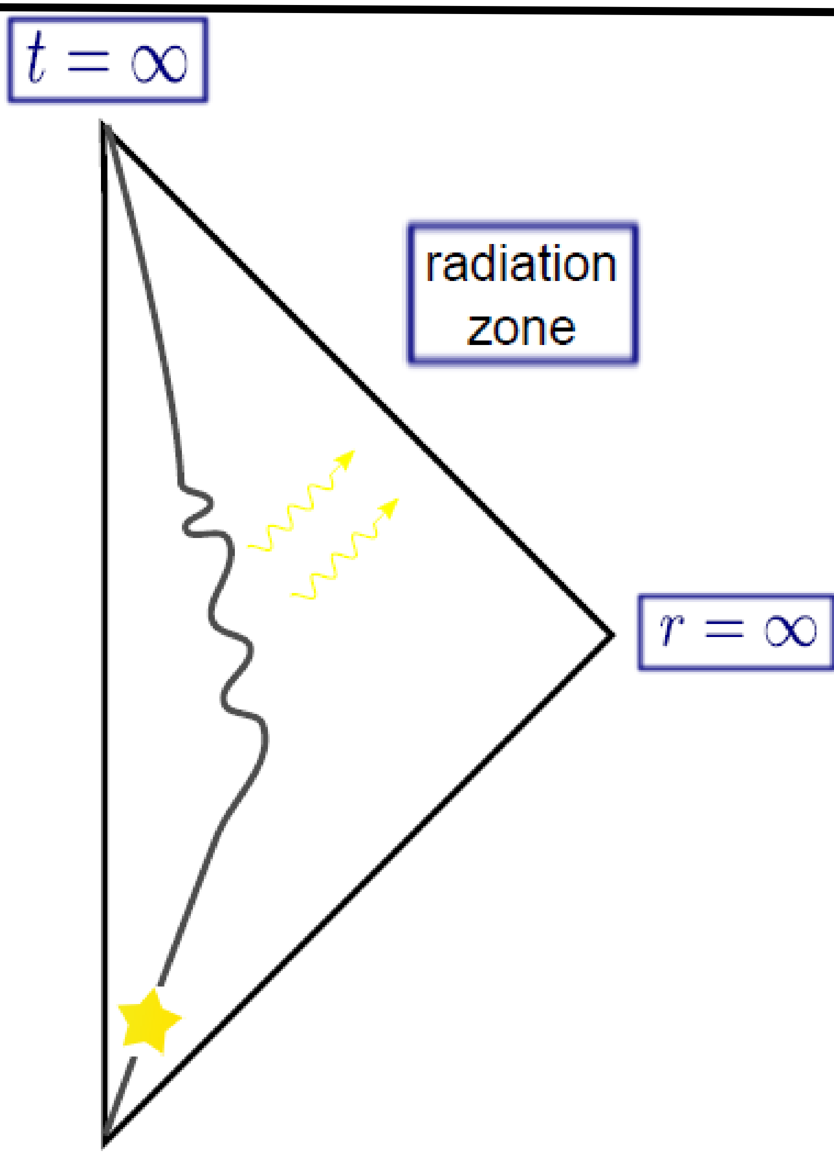
to ∞
points at 'infinity' where tracks meet

Conformal diagrams for flat spacetimes

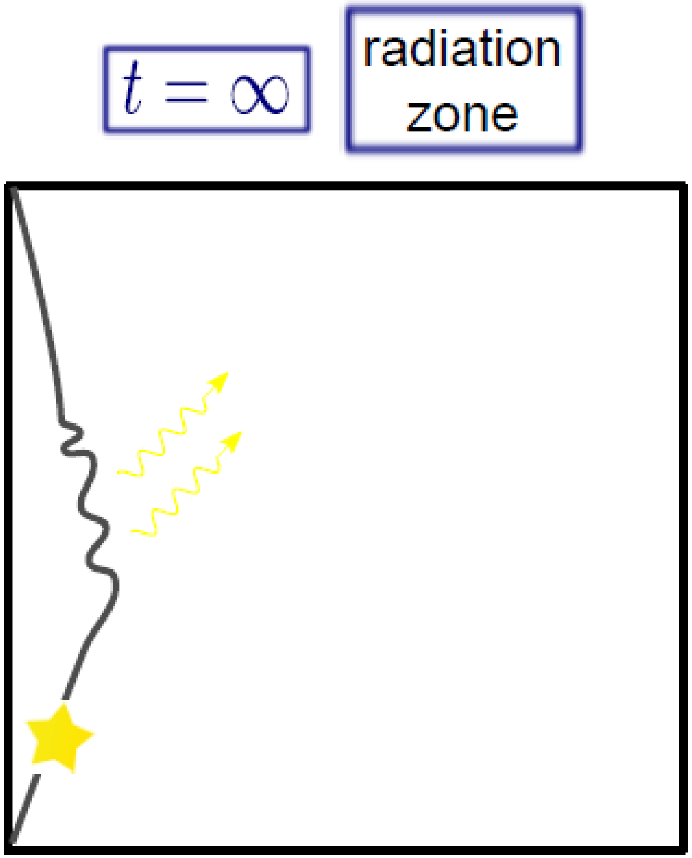
$\Lambda = 0$



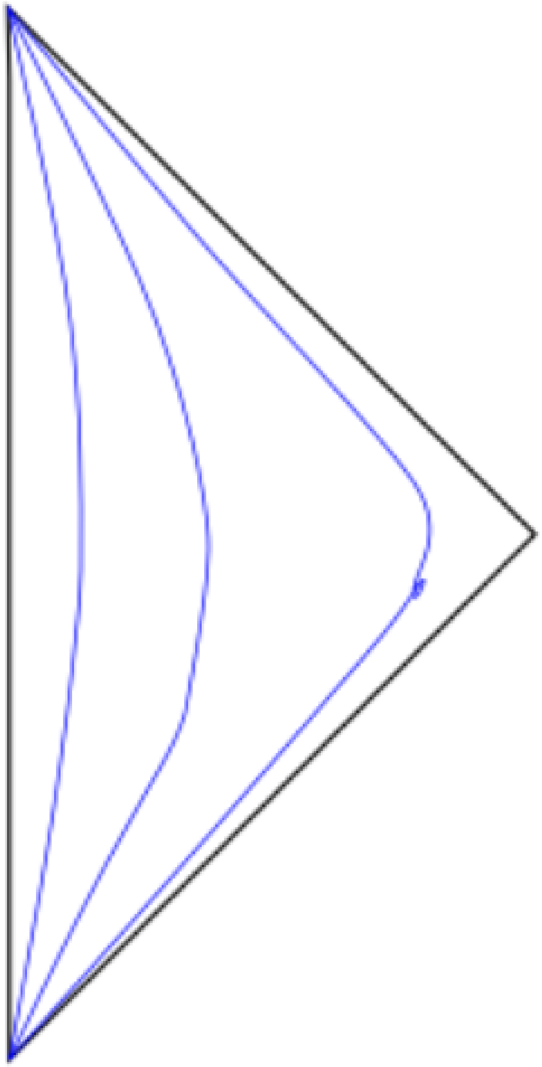
$\Lambda = 0$

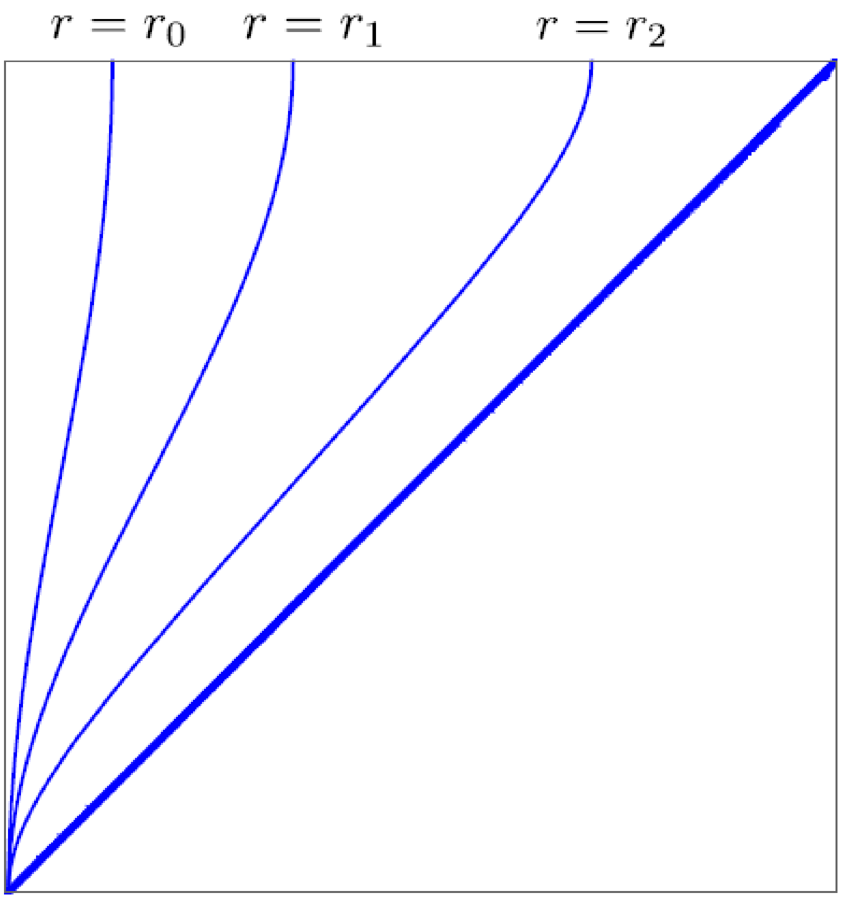


$\Lambda > 0$



$1/r$-expansion not applicable when $\Lambda \neq 0$





3 ingredients for the quadrupole formula

- Gravitational perturbation
- Quadrupole moment
- Conservation stress-energy tensor

First ingredient: gravitational perturbations

$$ds^2 = -dt^2 + e^{2\sqrt{\frac{\Lambda}{3}}t}\left(dx^2 + dy^2 + dz^2\right)$$

Gravitational perturbation satisfies

$$\left(-\frac{\partial^2}{\partial t^2} + e^{-2\sqrt{\frac{\Lambda}{3}}t}\vec{\nabla}^2 - 3\sqrt{\frac{\Lambda}{3}}\frac{\partial}{\partial t}\right)\bar{h}_{ij} = 16\pi G\, e^{-2\sqrt{\frac{\Lambda}{3}}t}\, T_{ij}$$

so that in the late time regime

$$\bar{h}_{ij} = \frac{4G}{r}\int \mathrm{d}^3x'\, T_{ij}(t_r, x') \underbrace{-G\sqrt{\frac{16\Lambda}{3}}\int_{-\infty}^{t_r}\mathrm{d}t' e^{\sqrt{\frac{\Lambda}{3}}t'}\frac{\partial}{\partial t'}\int \mathrm{d}^3x'\, T_{ij}(t', x')}_{\text{tail term}}$$

Second ingredient: quadrupole moment

$$Q_{ij} := \underbrace{\int \mathrm{d}^3 V}_{a^3 \mathrm{d}x \mathrm{d}y \mathrm{d}z} \underbrace{\rho}_{T_{\mu\nu}\partial_t^\mu \partial_t^\nu} \underbrace{(a\, x_i)\,(a\, x_j)}_{\text{physical distance}}$$

Third ingredient: binding agent

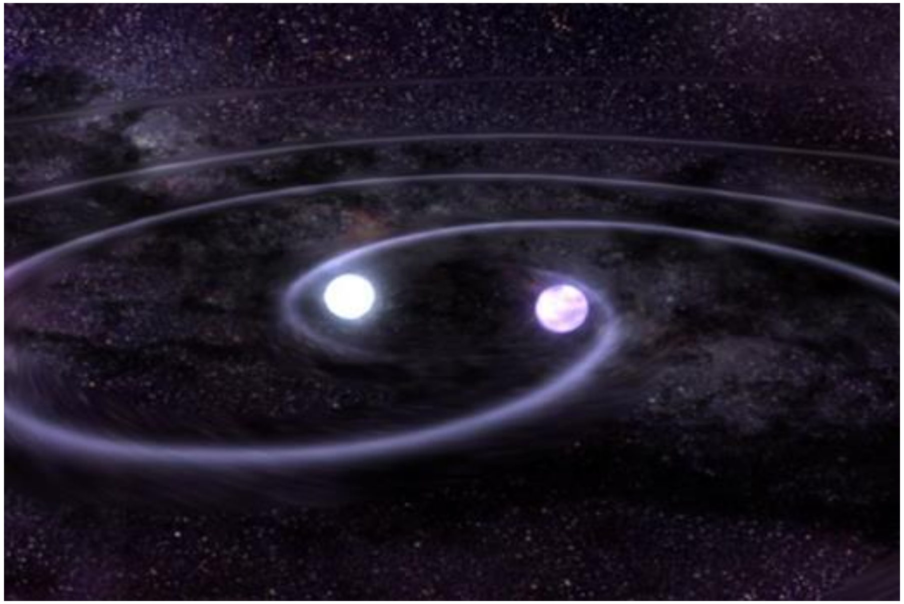
Conservation of the stress-energy tensor $\bar{\nabla}^\mu T_{\mu\nu} = 0$

$$\partial_t \rho - e^{-\sqrt{\frac{\Lambda}{3}}t} \vec{\nabla}^i T_{0i} + \sqrt{3\Lambda}\,(\rho + P) = 0$$

$$\partial_t T_{0i} - \vec{\nabla}^j T_{ij} + \sqrt{3\Lambda}\, T_{0i} = 0$$

$$\int \mathrm{d}^3x\, T_{ij} = \frac{e^{-\sqrt{\frac{\Lambda}{3}}t}}{2} \left(\ddot{Q}_{ij}^{(\rho)} + 2\sqrt{\frac{\Lambda}{3}}\dot{Q}_{ij}^{(\rho)} + \sqrt{\frac{\Lambda}{3}}\dot{Q}_{ij}^{(P)} + \frac{2\Lambda}{3} Q_{ij}^{(P)} \right)$$

Einstein's celebrated quadrupole formula



$$P_{\mathrm{Mink}} \hat{=} \frac{G}{8\pi} \int_{\mathcal{I}} d^2 S \; \dddot{Q}^{\mathrm{tt}}_{ab}(t_{\mathrm{ret}}) \dddot{Q}_{ab}(t_{\mathrm{ret}})$$

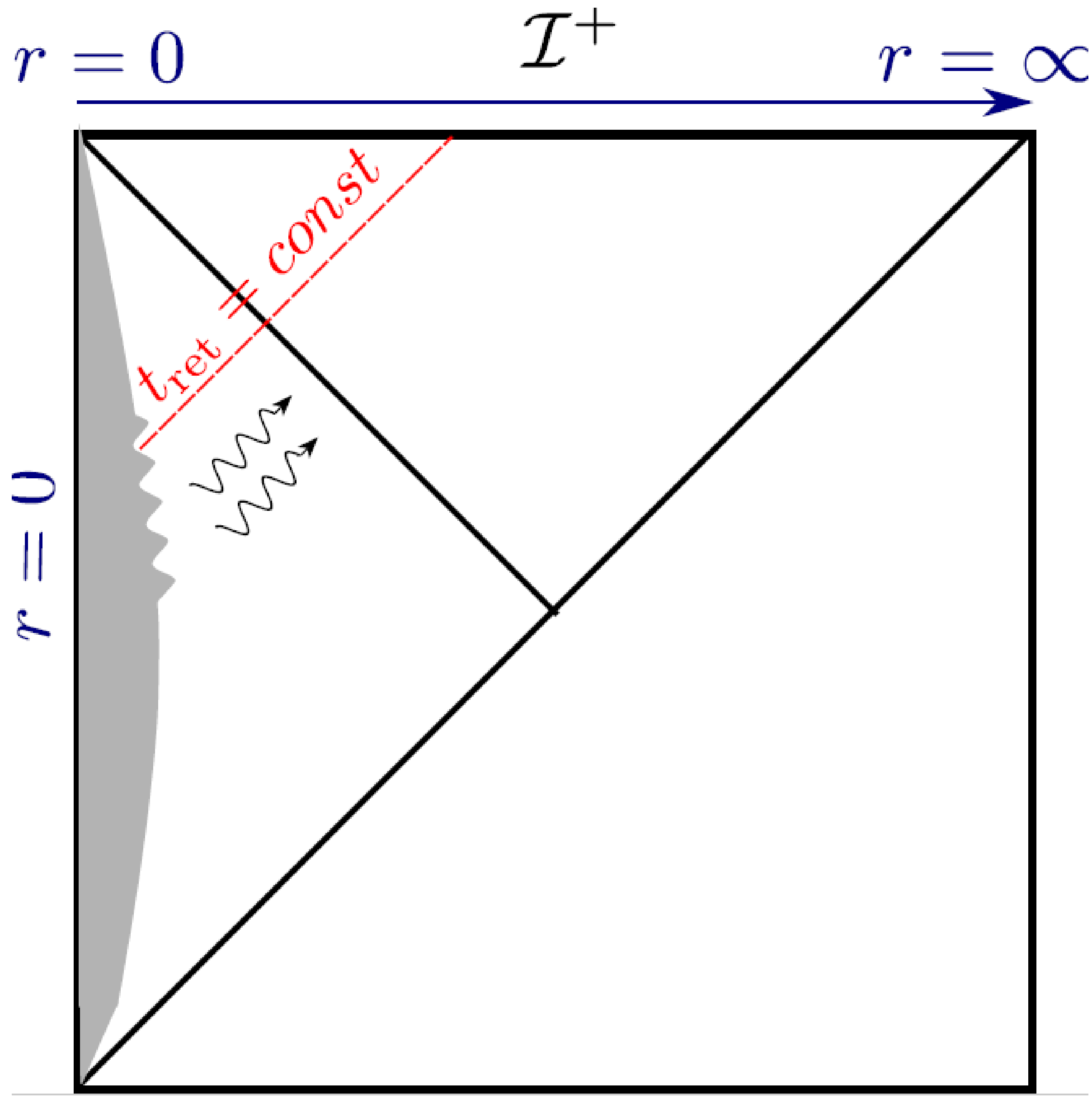
[Einstein, 1916]

… now with Λ!

$$P_{\mathrm{deSitter}} = \frac{G}{8\pi}\int_{\mathcal{I}} \mathrm{d}^2 S\, \mathcal{R}^{\mathrm{TT}}_{ab}(t_{\mathrm{ret}})\, \mathcal{R}^{ab}(t_{\mathrm{ret}})$$

$$\mathcal{R}_{ab} = \dddot{Q}^{(\rho)}_{ab} + \sqrt{3\Lambda}\,\ddot{Q}^{(\rho)}_{ab} + \frac{2\Lambda}{3}\dot{Q}^{(\rho)}_{ab} + \sqrt{\frac{\Lambda}{3}}\,\ddot{Q}^{(p)}_{ab} + \Lambda\dot{Q}^{(p)}_{ab} + 2\left(\frac{\Lambda}{3}\right)^{3/2} Q^{(p)}_{ab}$$

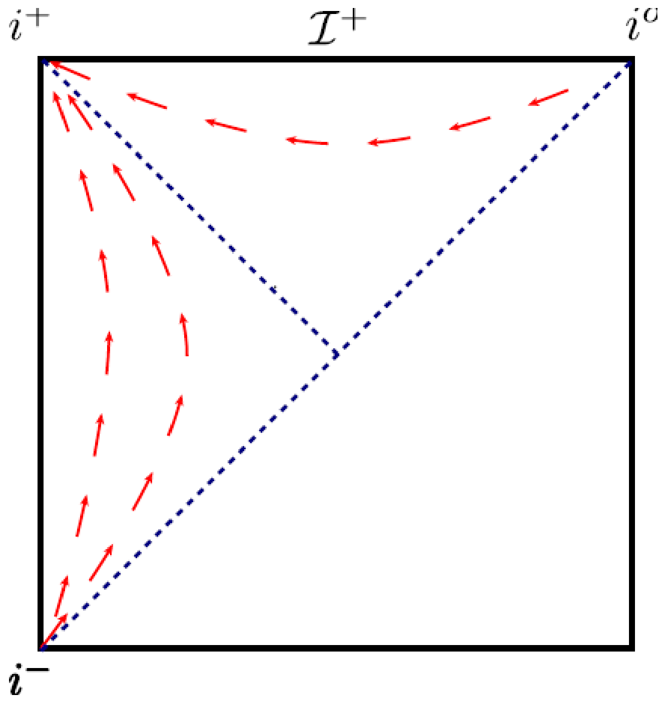
- Limit $\Lambda \to 0$ recovers Minkowski result
- Pressure terms appear
- Only retarded fields contribute despite tail term



$r = 0$
$\mathcal{I}^+$
$r = \infty$
$t_{\mathrm{ret}} = const$
$r = 0$

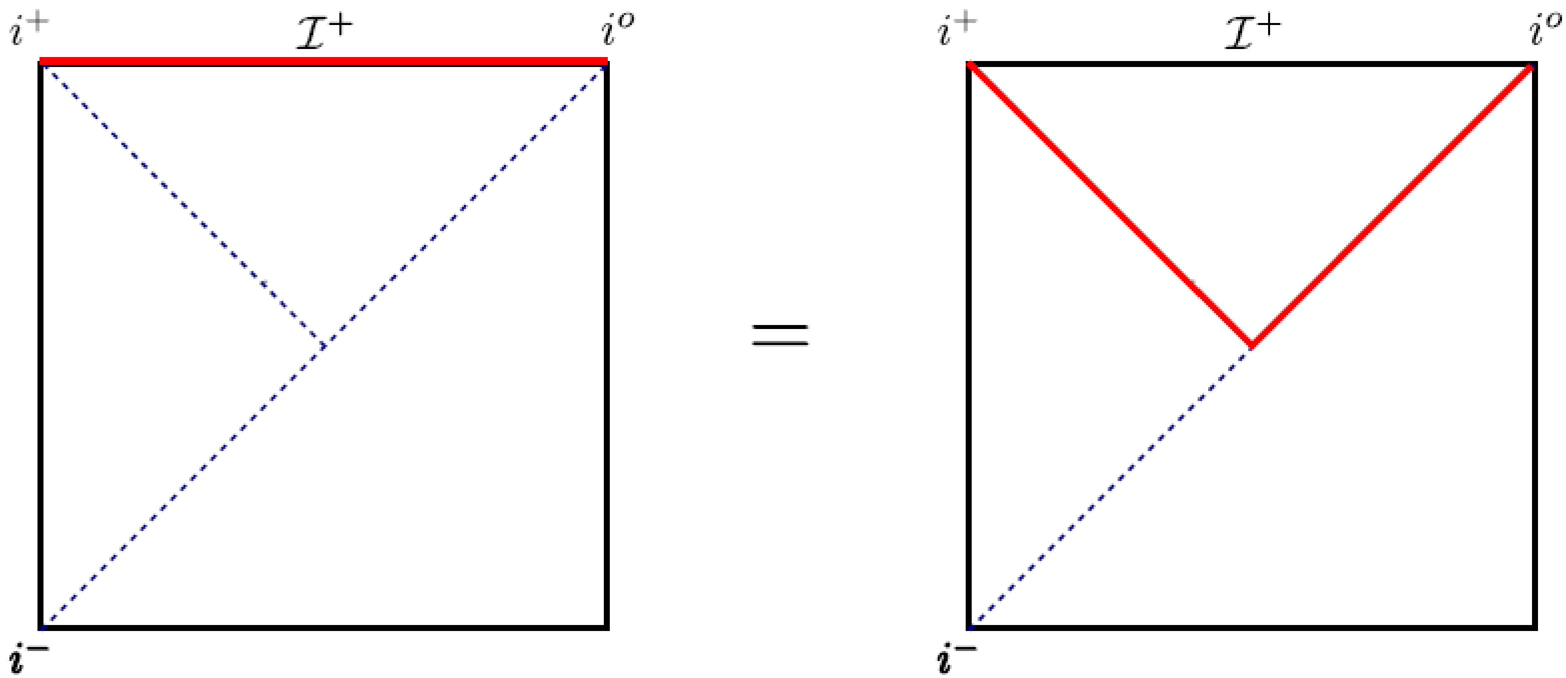
Is this power radiated positive?

All Killing vector fields are spacelike on $\mathcal{I}^+$



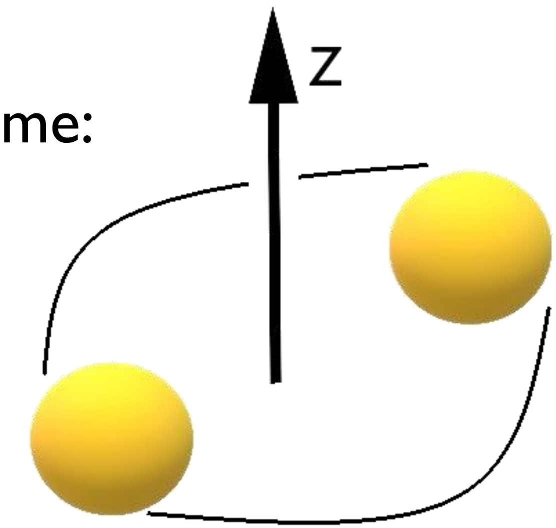
energy can be negative

For physically realistic sources it is!



Example: binary system

Assume system is in a circular orbit for all time:



$$\frac{dR_*}{dt} = 0 \text{ and } \frac{d\Omega_*}{dt} = 0$$

- Adiabatic approximation valid on cosmological time scales
- Fine-tuned trajectory
- System remains bound despite cosmological expansion

Example: binary system

$$P \hat{=} \frac{32G}{5}\mu^2 R_*^4 \Omega^6 \left(1 + \frac{5}{12}\frac{\Lambda}{\Omega^2} + \frac{1}{36}\frac{\Lambda^2}{\Omega^4}\right)$$

- Corrections $\sim \Lambda\, t_c^2$
- No truncation in Λ

Are these differences observable?

What systems would lead to 1% corrections to the power?

Assume that

- Corrections ~ $\sqrt{\Lambda}\, t_c$ generalizes to ~ $\sqrt{\mathrm{G}\,\rho(z)}\; t_c$

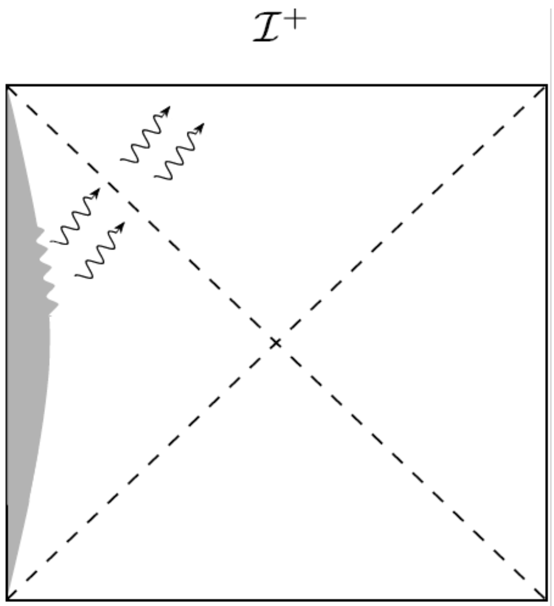
For z=100, characteristic time scale needs to be $t_c \sim 10^7$years

- $2\ M_{\odot}$ black hole binary: $d\ \sim\ 0.4\ pc$
- $10^6\ M_{\odot}$ black hole binary: $d\ \sim\ 30\ pc$

Conclusion

Even a tiny cosmological constant can cast a long shadow.

- Radiation zone very different
- $1/r$ expansions not useful
- Tail terms
- Potentially observable



Thank you for being here today!